Exercise 36

Find equations of the tangent lines to the curve

$$y = \frac{2}{1-3x}$$

at the points with x-coordinates 0 and -1.

Solution

Start by finding the points on the curve with these x-coordinates.

$$x = 0: \qquad y = \frac{2}{1-3(0)} = 2 \qquad \Rightarrow \qquad (0,2)$$

$$x = -1: \qquad y = \frac{2}{1-3(-1)} = \frac{1}{2} \qquad \Rightarrow \qquad \left(-1, \frac{1}{2}\right)$$

The slopes of the tangent lines to $y = 2/(1-3x)$ at the points, $(0,2)$ and $\left(-1, \frac{1}{2}\right)$, are found by calculating the derivative of $y = f(x)$ and then setting $x = 0$ and $x = -1$, respectively. Use the definition of $f'(x)$.

$$\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\
&= \lim_{h\to 0} \frac{\frac{2}{1-3(x+h)} - \frac{2}{1-3x}}{h} \\
&= \lim_{h\to 0} \frac{\frac{2}{1-3x-3h} - \frac{2}{1-3x}}{h} \\
&= \lim_{h\to 0} \frac{\frac{2(1-3x)}{(1-3x)(1-3x-3h)} - \frac{2(1-3x-3h)}{(1-3x)(1-3x-3h)}}{h} \\
&= \lim_{h\to 0} \frac{\frac{2(1-3x)-2(1-3x-3h)}{(1-3x)(1-3x-3h)}}{h} \\
&= \lim_{h\to 0} \frac{2(1-3x) - 2(1-3x-3h)}{h(1-3x)(1-3x-3h)} \\
&= \lim_{h\to 0} \frac{(2-6x) - (2-6x-6h)}{h(1-3x)(1-3x-3h)} \\
&= \lim_{h\to 0} \frac{6h}{h(1-3x)(1-3x-3h)} \\
&= \lim_{h\to 0} \frac{6}{(1-3x)(1-3x-3h)}
\end{aligned}$$

Evaluate the limit by setting $h = 0$.

$$f'(x) = \frac{6}{(1-3x)^2}$$

Therefore, the slopes at (0,2) and $\left(-1, \frac{1}{2}\right)$ are, respectively,

$$f'(0) = \frac{6}{(1)^2} = 6 \qquad \text{and} \qquad f'(-1) = \frac{6}{(1+3)^2} = \frac{3}{8}.$$

To determine the equations of the lines, use the points, these slopes, and the point-slope formula.

$$y - 2 = 6(x-0) \qquad\qquad y - \frac{1}{2} = \frac{3}{8}(x+1)$$

$$y - 2 = 6x \qquad\qquad y - \frac{1}{2} = \frac{3}{8}x + \frac{3}{8}$$

$$y = 6x + 2 \qquad\qquad y = \frac{3}{8}x + \frac{7}{8}$$

Below is a graph of $y = 6x + 2$, $y = \frac{3}{8}x + \frac{7}{8}$, and $y = \frac{2}{1-3x}$ versus x. Notice that the lines are tangent to the curve at $x = -1$ and $x = 0$.